Chain Rule: $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$

if $f(a)$ in the interior of domain of g

and both g and f differentiable

skip rest of proof.

Problem: let $f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

(a) is f continuous at $x=0$?

(b) is f differentiable at $x=0$?

Solution (a) to show: $\lim_{x \to 0} f(x) = f(0) = 0$!

Observe: $|f(x) - f(0)| = |x \sin \frac{1}{x} - 0|$

$= |x| \underbrace{|\sin \frac{1}{x}|}_{\leq 1} \leq |x|$

$\Rightarrow 0 \leq \lim_{x \to 0} |f(x) - f(0)| \leq \lim_{x \to 0} |x| = 0 \Rightarrow \lim_{x \to 0} f(x) = f(0)$

(b) need to show: $\lim\limits_{x\to 0} \dfrac{f(x)-f(0)}{x-0}$ does exist!

$$\frac{f(x)-\overset{0}{\overset{\shortparallel}{f(0)}}}{x-0} = \frac{x \sin \frac{1}{x}}{x} = \sin \frac{1}{x}$$

$\Rightarrow$ does $\boxed{\lim\limits_{x\to 0} \sin \frac{1}{x} \text{ exist?}}$

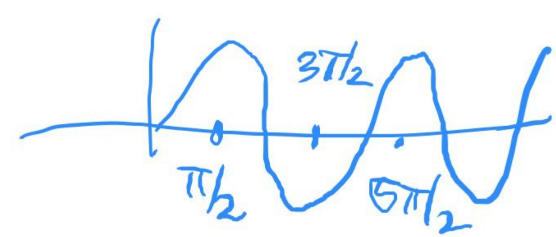
want to show: NO

consider sequence $\boxed{(x_n) = \left(\dfrac{1}{(n+\frac{1}{2})\pi}\right)_n \longrightarrow 0}$

$$\Rightarrow \sin \frac{1}{x_n} = \sin (n+\tfrac{1}{2})\pi = (-1)^n$$

$$\Rightarrow \lim_{n\to\infty} \frac{f(x_n)-f(0)}{x_n - 0} = \lim_{n\to\infty} \sin \frac{1}{x_n} = \lim_{n\to\infty} (-1)^n \quad \lightning$$

limit does not exist



$\Rightarrow$ f NOT differentiable at $x=0$.

Chapter 29 Mean Value Theorem

Theorem (Max/Min Test)

Assume I open interval $x_0 \in I$, $f: I \to \mathbb{R}$ differentiable

either $f(x_0) \geq f(x) \ \forall x \in I$ (local max.)

or $f(x_0) \leq f(x) \ \forall x \in I$ (local min)

$\Rightarrow$ $f'(x_0) = 0$

Proof (for x_0 max. i.e. $f(x_0) \geq f(x) \ \forall x \in I$)

$$f'(x_0) = \lim_{\substack{x \to x_0 \\ x > x_0}} \frac{\overbrace{f(x) - f(x_0)}^{\leq 0}}{\underbrace{x - x_0}_{>0}} \leq 0$$

Similarly:

$$f'(x_0) = \lim_{\substack{x \to x_0 \\ x < x_0}} \frac{\overbrace{f(x)-f(x_0)}^{\leq 0}}{\underbrace{x-x_0}_{< 0}} \geq 0$$

because $f(x_0) \geq f(x)$ local max.

$$\Rightarrow f'(x_0) = 0$$

<u>Rolle's Theorem</u> $\quad f: [a,b] \to \mathbb{R}$ continuous

$f: (a,b) \to \mathbb{R}$ differentiable

If $f(a) = f(b) \Rightarrow \exists\, x_0 \in (a,b)$ s.t. $f'(x_0) = 0$

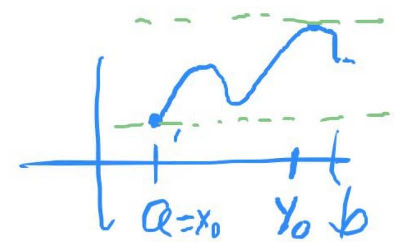
Proof. f cont. on $[a,b] \Rightarrow \exists\, x_0, y_0 \in [a,b]$ s.t.

$f(x_0) \leq f(x) \leq f(y_0) \quad \forall x \in [a,b]$

<u>case 1</u> assume $x_0 \in (a,b)$ or $y_0 \in (a,b)$

(here: take y_0)

by previous theorem: $f'(y_0) = 0$ ✓



only remaining

Case 2 both x_0 and y_0 at boundary (i.e. $= a$ or $= b$)

$\Rightarrow \quad f(x_0) \underset{\text{say}}{=} f(a) \underset{\text{assumption}}{=} f(b) = f(y_0)$

$\Rightarrow \quad f(x_0) \leq f(x) \leq f(y_0) = f(x_0) \quad \forall x$

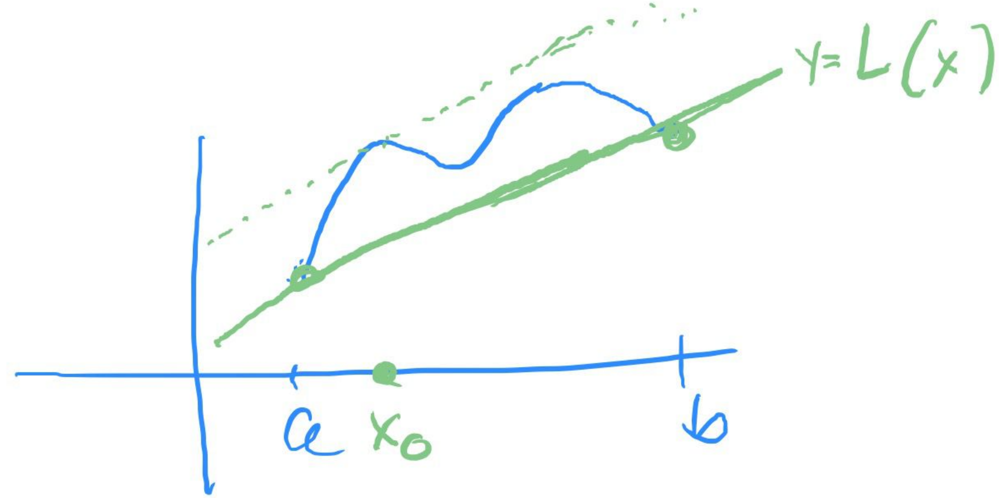
$\Rightarrow$ constant function $\Rightarrow f'(x) = 0 \ \forall x \in (a,b)$. ✓

Mean Value Theorem $\quad f : [a,b] \to \mathbb{R}$ cont.
$\quad f : (a,b) \to \mathbb{R}$ differentiable

$\Rightarrow \exists$ at least one $x_0 \in (a,b)$ s.t.

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}$$

Proof



$L(x)$ given by

Line through end points

$$\text{slope} = \frac{f(b)-f(a)}{b-a}$$

$$L(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$

(check: $L(a) = f(a)$
$L(b) = f(b)$)

define $g(x) = f(x) - L(x)$

$$\Longrightarrow \quad g(a) = 0 = g(b)$$

Rolle's Theorem $\Rightarrow$ $\exists\ x_0 \in (a,b)$:

$$g'(x_0) = 0$$
$$\shortparallel$$
$$f'(x_0) - \underbrace{\frac{f(b)-f(a)}{b-a}}_{= L'(x_0)} = 0$$

$\Rightarrow$ claim.

Corollary 1

$f : (a,b) \to \mathbb{R}$ differentiable

assume $f'(x) = 0 \ \forall x \in (a,b) \Rightarrow f(x) = c$ constant number for all $x \in (a,b)$

proof. enought to show: $f(x_1) = f(x_2) \ \forall x_1, x_2 \in (a,b)$
say $x_1 < x_2$.

by Mean Value Theorem:

$$\exists x_0 \in (x_1, x_2) : \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x_0) = 0$$

$(a = x_1, \ b = x_2)$

$$\Rightarrow f(x_2) - f(x_1) = 0 \rightarrow \text{claim} \checkmark$$

Corollary 2 $\quad f, g : (a,b) \to \mathbb{R}$ differentiable

$$f'(x) = g'(x) \quad \forall x \in (a,b)$$

$\Rightarrow \exists c$ constant s.t. $f(x) = g(x) + c$ for all $x \in (a,b)$

proof $\quad$ apply Corollary 1 to function $f(x) - g(x)$

$$(f-g)'(x) = f'(x) - g'(x) = 0$$

$$\overset{\text{Cor. 1}}{\Rightarrow} \quad f(x) - g(x) = c \quad \text{for some constant } c$$

Intermediate Value Theorem for derivatives

$f : (a,b) \to$ differentiable

If $\quad a < x_1 < x_2 < b$ and c a number between $f'(x_1)$ and $f'(x_2)$

$\Rightarrow \exists x \in (x_1, x_2)$ s.t. $f'(x) = c$